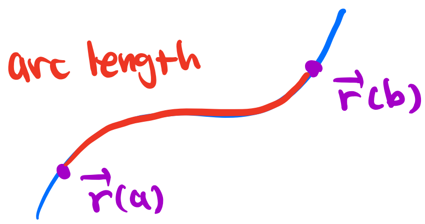
13.3. Arclength and curvature

Prop Given a curve parametrized by $\vec{r}(t)$, the arc length between $\vec{r}(a)$ and $\vec{r}(b)$ is equal to

$$\int_a^b |\vec{r}\,'(t)|\,dt.$$



* Physical interpretation:

$|\vec{r}\,'(t)|$ = speed at time t

$\Rightarrow \int_a^b |\vec{r}\,'(t)|\,dt$ = distance traveled on $a \le t \le b$.

Def Consider a vector function $\vec{r}(t)$.

(1) Its arc length parameter at t is

$$s := \int_0^t |\vec{r}\,'(u)|\,du$$ = arc length from $\vec{r}(0)$ to $\vec{r}(t)$

(2) Its curvature at t is

$$K(t) := \boxed{\frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

On the formula sheet

Note The curvature measures how fast the curve changes its direction at each point.

e.g. lines have curvature 0.

circles of radius R have curvature $\frac{1}{R}$.

Ex Consider the curve C parametrized by

$$\vec{r}(t) = (4\cos t,\ 4\sin t,\ 3t), \quad 0 \le t \le 6$$

(1) Find the length of C.

Sol $\text{Length} = \int_0^6 |\vec{r}'(t)|\, dt.$

$$\vec{r}'(t) = (-4\sin t,\ 4\cos t,\ 3)$$

$$|\vec{r}'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5.$$

$$\Rightarrow \text{Length} = \int_0^6 5\, dt = \boxed{30}$$

(2) Find the curvature of C.

Sol $K(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

$$\vec{r}''(t) = (-4\cos t,\ -4\sin t,\ 0)$$

$$\vec{r}'(t) \times \vec{r}''(t) = (12\sin t,\ -12\cos t,\ 16)$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{144\sin^2 t + 144\cos^2 t + 256} = 20.$$

$$\Rightarrow K(t) = \frac{20}{5^3} = \boxed{\frac{4}{25}}$$

(3) Parametrize C by arc length.

Sol Idea: Write t in terms of the arc length parameter s.

$$s = \int_0^t |\vec{r}'(u)|\,du = \int_0^t 5\,du = 5t$$

$$\Rightarrow t = s/5.$$

$$\Rightarrow \begin{cases} x = 4\cos t = 4\cos(s/5) \\ y = 4\sin t = 4\sin(s/5) \\ z = 3t = 3s/5. \end{cases}$$

Domain: $0 \le t \le 6 \Rightarrow 0 \le s \le 30$ ($\because s = 5t$)

$\rightsquigarrow \vec{r}_1(s) = \boxed{(4\cos(s/5),\ 4\sin(s/5),\ 3s/5) \text{ with } 0 \le s \le 30.}$

(4) Find the midpoint of C.

Sol Since the length of C is 30,

the arc length parameter at the midpoint is $s = \frac{1}{2} \cdot 30 = 15$.

$\rightsquigarrow \vec{r}_1(15) = \boxed{(4\cos(3),\ 4\sin(3),\ 9)}$

Ex Let C be the curve parametrized by

$$\vec{r}(t) = \left(\cos(3t+3),\ \sin(3t+3),\ 2t^{3/2} \right),\quad t \geq 0.$$

Parametrize C by arc length.

Sol $\vec{r}\,'(t) = \left(-3\sin(3t+3),\ 3\cos(3t+3),\ 3t^{1/2}\right).$

$$|\vec{r}\,'(t)| = \sqrt{9\sin^2(3t+3) + 9\cos^2(3t+3) + 9t}$$

$$= \sqrt{9+9t} = 3\sqrt{1+t}.$$

$$\Rightarrow s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t 3\sqrt{1+u}\,du$$

$$= 2(1+u)^{3/2}\Big|_{u=0}^{u=t} = 2(1+t)^{3/2} - 2.$$

$$\rightsquigarrow \frac{s}{2} + 1 = (1+t)^{3/2} \rightsquigarrow \left(\frac{s}{2}+1\right)^{2/3} = 1+t$$

$$\Rightarrow t = \left(\frac{s}{2}+1\right)^{2/3} - 1.$$

$$\Rightarrow \begin{cases} x = \cos(3t+3) = \cos\left(3\left(\frac{s}{2}+1\right)^{3/2}\right) \\ y = \sin(3t+3) = \sin\left(3\left(\frac{s}{2}+1\right)^{3/2}\right) \\ z = 2t^{2/3} = 2\left(\left(\frac{s}{2}+1\right)^{3/2} - 1\right)^{2/3} \end{cases}$$

$$\rightsquigarrow \boxed{\vec{r}_1(s) = \left(\cos\left(3\left(\frac{s}{2}+1\right)^{3/2}\right),\ \sin\left(3\left(\frac{s}{2}+1\right)^{3/2}\right),\ 2\left(\left(\frac{s}{2}+1\right)^{3/2}-1\right)^{2/3}\right) \text{ with } s \geq 0}$$